Math 210A Lecture 25 Notes

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1 Localization of Rings

1.1 Construction and properties

Let's say we have a commutative ring where not every element has a multiplicative inverse. How do we add in more elements to get a larger ring with some more inverses? We may not want to add in all inverses if we want to preserve the structure of the original ring.

Definition 1.1. A subset S of a ring R is **multiplicatively closed** if it closed under multiplication, $1 \in S$, and $0 \notin S$.

Lemma 1.1. *Suppose R is commutative, and let $S \subseteq R$ be multiplicatively closed. The relation $\sim$ on $R \times S$ given by $(a, s) \sim (b, t)$ iff there exists $r \in S$ such that $rat = rbs$ is an equivalence rrelation.*

Proof. Let's verify transitivity. Suppose $(a, s) \sim (b, t)$ and $(b, t) \sim (c, u)$, so there exist $r, q \in S$ such that $rat = rbs$ and $qbu = qct$. Note that $rqt \in S$ since these elements are all in S. Then

$$(rqt)au = q(rat)u = q(rbs)u = rs(qbu) = rs(qct) = (rqt)cs. \qquad \square$$

Remark 1.1. If S contains no zero divisors, then $rat = rbs \implies at = bs$. So we can replace the condition in $\sim$ with $at = bs$. his of this as $a/s = b/t$.

Definition 1.2. The equivalence class of (a, s) under $\sim$ is denoted a/s (or $\frac{a}{s}$) The set of equivalence classes is $S^{-1}R$.

Theorem 1.1. *$S^{-1}R$ is a commutative ring with addition and multiplication*

$$\frac{a}{s} + \frac{b}{t} = \frac{at + bs}{st}, \qquad \frac{a}{s} \cdot \frac{b}{t} = \frac{ab}{st}.$$

Proof. If $(a,s) \sim (a',s')$, we want that $(at+bs)/(st) = (a't+bs')(s't)$. There exists $r \in S$ such that $ras' = ra's$. Then

$$r(st+bs)s' = rats' + rbss' = ras'ts + rbss' = r(a't+bss')s.$$

Showing multiplication is well-defined is similar (and a bit easier, actually). The additive identity is $0/1$, and the multiplicative identity is $1/1$. $\square$

Remark 1.2. We have a homomorphism $\iota : R \to S^{-1}R$ with $\iota(r) = r/1$. This is injective iff S has no zero divisors. $a/1 = b/1$ means $ra = rb$. This means $ra = rb$ if S has no zero divisors, and otherwise, we can find a, b, r such that $ra = rb$ but $a \neq b$.

Remark 1.3. If $s \in S$ is a zero divisor and $rs = 0$ with $r \in R$. then $0 = 0/s = rs/s = r/1 = s$, so $r \mapsto 0$ in $S^{-1}R$. But S maps into $(S^{-1}R)^\times$ because $s \cdot 1/s = 1$. Also, no elements get mapped to 0 because if $s/1 = 0/1 = 0$, then there exists some $r \in S$ with $rs = 0$, which is impossible because $0 \notin S$.

Remark 1.4. If $\phi : R \to R'$ is a ring homomorphism such that $\phi(S) \subseteq (R')^\times$, then there exists a unique homomorphism $\psi : S^{-1}R \to R'$ such that

$$\begin{tikzcd} R \arrow[r, "\iota"] \arrow[dr, "\phi"'] & S^{-1}R \arrow[d, "\psi"] \\ & R' \end{tikzcd}$$

given by $\psi(a/s) = \phi(a)\phi(s)^{-1}$.

1.2 Examples of localizations

Definition 1.3. Let R be a domain and $S = R \setminus \{0\}$ then $Q(R) := S^{-1}R$ is the **fraction field**, field of fractions, or **quotient field** of R. It is the "smallest" field containing R.

Example 1.1. Let F be a field. $Q(F[x]) = F(x)$, the field of rational functions over F. These are $f(x)/g(x)$ where $g \neq 0$.

Definition 1.4. Let $S = T \setminus (\{\text{zero divisors}\} \cup \{0\})$. Then $Q(R) = S^{-1}R$ is called the **total ring of fractions**.

Example 1.2. Let $R = \mathbb{Z} \times \mathbb{Z}$. You can check that $Q(R) = \mathbb{Q} \times \mathbb{Q}$. In fact, if $R = R_1 \times R_2$, then $Q(R) = Q(R_1) \times Q(R_2)$.

Example 1.3. Let $R = F[x,y]/(xy)$, and $S = \{x^n : n \geq 0\}$. Then $S^{-1}R \cong F[x, x^{-1}]$, via the isomorphism $x \mapsto x$ and $y \mapsto 0$.

Definition 1.5. Let $S_p = S \setminus p$, where p is a prime ideal. The ring $R_p = S^{-1}pR$. is the **localization of** R at p.

Note that $pR_p \subseteq R_p$, so $(R_p)^\times = R_p \setminus pR_p$. So pRp is the unique maximal ideal in R_p.

Definition 1.6. A commutative ring with a unique maximal ideal is called a **local ring**.

Example 1.4. Let $p \in \mathbb{Z}$ be prime. Then $\mathbb{Z}_{(p)} = \{a/b \in Q : p \nmid b\}$.

Example 1.5. $F[x]_{(x)} = \{f/g : x \nmid g\}$.

Example 1.6. Let $x \in R$ be not a zero-divisor. Let $S = \{x^n : n \geq 0\}$. Then $R_x = S^{-1}R = \{a/x^n : a \in R, n \geq 0\}$. If $R = F[x]$ and $x = x$, then $F[x]_x = F[x, x^{-1}] \subseteq F(x)$.

Proposition 1.1. *Let $\iota : R \to S^{-1}R$ send $r \mapsto r/1$. Let $I \subseteq R$ be an ideal.*

1. *$S^{-1}I = \{a/s : a \in I, s \in S\}$ is an ideal of $S^{-1}R$.*
2. *$\iota^{-1}(S^{-1}I) = \{a \in R : Sa \cap I \neq \varnothing\}$.*
3. *If $J \subseteq S^{-1}R$, then $S^{-1} \cdot \iota^{-1}(J) = J$.*

Proof. For part 1, $a/s + b/t = (at + bs)/(st)$, where $at + bs \in I$ and $st \in S$. Then $r \cdot a/s = (ra)/s$, where $r \in I$ and $s \in S$.

For part 2, let $\phi(s) = b/s$, where $b \in I$, $s \; inS$, and $a \in R$. What properties must a have? Then $a/1 = b/s$ iff there eixsts some $r \in S$ such that $ras = rb$. This is true for some b, s iff there exists some $r' \in S$ such that $r'a \in I$.

The proof of part 3 is left as an exercise. □